Online Appendix

to accompany

"Coordination in the Fight Against Collusion"

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In this Online Appendix, we provide in Section OA-A the proofs of the auxiliary lemmas used in the main Appendix. We then address three points raised in the paper. First, in Section OA-B, we provide an extension of the model to allow more than two markets, as mentioned in footnote 13 in the paper. Second, in Section OA-C, we show that our monotonicity assumptions are satisfied when costs are distributed according to the power distribution, as mentioned in footnote 35 in the paper. Third, in Section OA-D, we provide additional illustrations of coordination and coordination failure with independent buyers, as mentioned in footnote 40 in the paper.

OA-A Proofs of auxiliary lemmas

OA-A.1 Proof of Lemma A.1

We have: $B(r) + C(r) = \overline{\pi}^m(r) - \overline{\pi}^n(r) = G(r)\overline{\pi}^m(r)$, where $G(r)$ and $\overline{\pi}^m(r)$ are both positive for $r > \underline{c}$, and strictly increasing; it follows that $B(r) + C(r)$ is also positive and strictly increasing. Likewise, using $S(r) = \pi^m(\underline{c}) - \pi^n(\underline{c}) = G(r)\pi^m(\underline{c}; r)$, we have: $S(r) - B(r) - C(r) = G(r)\left[\pi^m(\underline{c}; r) - \overline{\pi}^m(r)\right]$, where $G(r)$ and

$$\pi^m(\underline{c}; r) - \overline{\pi}^m(r) = (r - \underline{c}) - \int_{\underline{c}}^{r} G(c)dc = \int_{\underline{c}}^{r} [1 - G(c)]\, dc$$

are both positive for $r > \underline{c}$, and strictly increasing. The conclusion follows. ■

OA-A.2 Proof of Lemma A.2

Part (i). Let $\Psi \equiv \left(1-\delta^2\right)\{[L(\overline{r},\underline{r},\delta)-S(\underline{r})]-[L(\underline{r},\overline{r},\delta)-S(\overline{r})]\}$. Straightforward manipulations yield:

$$\begin{aligned}\Psi &= \left\{\delta\left[B\left(\underline{r}\right)-C\left(\overline{r}\right)\right]+\delta^2\left[B\left(\overline{r}\right)-C\left(\underline{r}\right)\right]-\left(1-\delta^2\right)S\left(\underline{r}\right)\right\}\\ &\quad -\left\{\delta\left[B\left(\overline{r}\right)-C\left(\underline{r}\right)\right]+\delta^2\left[B\left(\underline{r}\right)-C\left(\overline{r}\right)\right]-\left(1-\delta^2\right)S\left(\overline{r}\right)\right\}\\ &= (1-\delta)\left\{(1+\delta)S\left(\overline{r}\right)-\delta\left[B\left(\overline{r}\right)+C\left(\overline{r}\right)\right]\right\}\\ &\quad -(1-\delta)\left\{(1+\delta)S\left(\underline{r}\right)-\delta\left[B\left(\underline{r}\right)+C\left(\underline{r}\right)\right]\right\}\\ &= (1-\delta)\left[\psi\left(\overline{r}\right)-\psi\left(\underline{r}\right)\right],\end{aligned}$$

where $\psi(r) \equiv (1+\delta)S(r)-\delta[B(r)+C(r)]$ is strictly increasing in r, that is $\psi'(r) = (1+\delta)S'(r)-\delta[B'(r)+C'(r)] > 0$, where the inequality follows from $\delta \geq 0$ and Lemma A.1. Therefore, $\Psi \geq 0$, implying that the more stringent condition in (6) is $L(\underline{r},\overline{r},\delta) \geq S(\overline{r})$.

Part (ii). We have:

$$\begin{aligned}\frac{\partial L\left(\underline{r},\overline{r},\delta\right)}{\partial\delta} &= \frac{2\delta^2}{\left(1-\delta^2\right)^2}\left\{B(\underline{r})-C(\overline{r})+\delta\left[B(\overline{r})-C(\underline{r})\right]\right\}\\ &\quad +\frac{1}{1-\delta^2}\left\{B(\underline{r})-C(\overline{r})+2\delta\left[B(\overline{r})-C(\underline{r})\right]\right\}\\ &= \frac{(1-\delta)^2\left[B(\underline{r})-C(\overline{r})\right]+2\delta\left[B(\underline{r})-C(\underline{r})+B(\overline{r})-C(\overline{r})\right]}{1-\delta^2}\\ &> 0,\end{aligned}$$

where the second equality rearranges, and the inequality follows from Lemma A.1 and $\underline{r} \leq \overline{r}$, which together imply $B(\underline{r}) > C(\underline{r})$ and $B(\underline{r}) \geq B(\overline{r}) > C(\overline{r})$. ■

OA-A.3 Proof of Lemma A.3

We first establish the existence and properties of the deterrence thresholds $r_S^D(\delta)$ and $r_U^D(\delta) \equiv \hat{\delta}_U^{-1}(\delta)$. From Assumption S and Lemma 3, $\hat{\delta}_S(r) = \hat{\delta}(r,r)$ is strictly decreasing in r and tends to 1 as r tends to $\underline{c}$. Hence, with symmetric reserves, collusion is an issue if $\delta > \hat{\delta}_S(\min\{\bar{c},v\})$, in which case setting the reserves to r deters it if and only if $r \leq r_S^D(\delta) \equiv \hat{\delta}_S^{-1}(\delta)$. Assumption S moreover ensures that $r_S^D(\delta)$ is strictly decreasing in δ.

Similarly, Assumption U and Lemma 3 together ensures that $\hat{\delta}_S(r)$ is also strictly decreasing in r, and tends to 1 as r tends to $\underline{c}$. Hence, in the case of a unique market, collusion is an issue if $\delta > \hat{\delta}_U(\min\{\bar{c}, v\})$, in which case setting the reserve equal to r deters it if and only if $r \leq r_U^D(\delta) \equiv \hat{\delta}_U^{-1}(\delta)$, where $r_U^D(\delta)$ is strictly decreasing in δ.

By construction, $\hat{\delta}\left(\underline{c}, r_U^D(\delta)\right) = \delta$ and $r_U^D(\delta) > \underline{c}$ (as $\hat{\delta}(\underline{c}, \underline{c}) = 1$). Hence, from Assumption L, $\hat{\delta}_S\left(r_U^D(\delta)\right) = \hat{\delta}\left(r_U^D(\delta), r_U^D(\delta)\right) < \delta$; that is, symmetric reserves equal to $r_U^D(\delta)$ would not deter collusion. It then follows from Assumption S that

$$r_U^D(\delta) > r_S^D(\delta).$$

The rest of the proof proceeds in three steps. We start by checking that any $\mathbf{r} \leq \mathbf{r}_S^D(\delta)$ deters collusion (step 1), before characterizing the other deterrence reserves (step 2), and establishing the monotonicity of $\mathcal{D}(\delta)$ in δ (step 3).

- *Step* 1. Fix $\mathbf{r} \leq \mathbf{r}_S^D(\delta)$, and let $\bar{r} = \max\{r_1, r_2\}$ denote the higher of the two reserves. Because $\mathbf{r}_S^D(\delta) \in (\delta)$ and $\bar{r} \leq r_S^D(\delta)$, it follows from Assumption S that $(\bar{r}, \bar{r}) \in \mathcal{D}(\delta)$. And because $\mathbf{r} \leq (\bar{r}, \bar{r})$, it follows from Assumption L that $\mathbf{r} \in \mathcal{D}(\delta)$.

- *Step* 2. Fix $r > r_U^D(\delta)$, which amounts to $\delta > \hat{\delta}_U(r)$; collusion would thus be sustainable if there were a unique market with reserve r. We thus have $S(r) < L(\underline{c}, r, \delta)$ and, from Assumption L, $S(r) < L(r', r, \delta)$ for any $r' \geq \underline{c}$; that is, collusion is sustainable, regardless of the reserve set in the other market. Hence, all deterrence reserves lie below $r_U^D(\delta)$.

Fix now $r \in (r_S^D(\delta), r_U^D(\delta)]$, implying that $(\underline{c}, r)$ deters collusion (i.e., $\hat{\delta}(\underline{c}, r) \geq \delta$) whereas (r, r) does not (i.e., $\hat{\delta}(r, r) < \delta$). From Assumption L, in the range $r' \leq r$, $\delta(r', r)$ is strictly decreasing in r'. Hence, for any δ, conditional on setting the reserve r in one market and a lower reserve $r' \leq r$ in the other market, collusion is deterred if and only if $r' \leq \hat{r}(r_j, \delta)$, where $\hat{r}(r, \delta)$ is the unique solution in r' to $\hat{\delta}(r', r) = \delta$ in the range $r' \leq r$. Assumption L moreover ensures that $\hat{r}(r, \delta)$ is strictly decreasing in δ.

- *Step* 3. As δ increases, the deterrence set $\mathcal{D}(\delta)$ shrinks: for any $\delta \in (0,1)$, $\mathbf{r} \in \mathcal{D}(\delta)$ amounts to $\hat{\delta}(\mathbf{r}) \geq \delta$, which in turn implies $\hat{\delta}(\mathbf{r}) > \delta'$ for any $\delta' < \delta$; hence, $\mathcal{D}(\delta) \subseteq \mathcal{D}(\delta')$ for any $\delta' < \delta$. Furthermore, $\mathcal{D}(\delta)$ is strictly shrinking as δ increases: as $\hat{r}(r, \delta)$ is strictly decreasing in δ, in the range $r_j \leq r_i$, any $\mathbf{r}$ such that $r_i \in \left[r_S^D(\delta), r_U^D(\delta)\right]$ and $r_j = \hat{r}(r_i, \delta)$ deters collusion for δ, but no longer does so for any $\delta' > \delta$. Finally, $\mathcal{D}(\delta)$ shrinks continuously as δ increases. In particular, for any δ and any $\mathbf{r} = (r_1, r_2) \in \mathcal{D}(\delta)$, there exists a nearby $\mathbf{r}'$ that belongs to $\mathcal{D}(\delta')$ for δ' higher but sufficiently close to δ: for instance, in the range $r_2 \leq r_1$, if $r_2 > \underline{c}$, then $\mathbf{r}' = (r_1, r_2')$ would do, for r_2' slightly below r_j; and if instead $r_2 = \underline{c}$, then $\mathbf{r}' = (r_1', r_2)$ would do, for r_1' slightly below r_1. ■

OA-A.4 Proof of Lemma A.4

That $U^D(\delta)$ is continuous follows directly from the Maximum Theorem, as the buyer's competitive payoff $\bar{U}^{Comp}(\mathbf{r})$is continuous in $(\mathbf{r})$ and the deterrence set is compact and continuous in δ.[1] That $U^D(\delta)$ is strictly decreasing follows from the fact that $\mathbf{r}^D(\delta)$ lies in the boundary of the deterrence set, which is strictly shrinking in δ.[2] By construction, $U^D\left(\delta^C\right) = U^{Comp}\left(r^C\right) = \max_r U^{Comp}(r) > U^{Comp}\left(r^A\right) > U^{Coll}\left(r^A\right)$. Finally, $\lim_{\delta\longrightarrow 1} U^D(\delta) = 0$, as the deterrence set converges to $\{(\underline{c},\underline{c})\}$ as δ tends to 1. ■

OA-A.5 Proof of Lemma A.5

We have: $\frac{\partial\bar{\phi}(r,\delta)}{\partial r} + \frac{\partial\underline{\phi}(r,\delta)}{\partial r} = (1-\delta)\left\{(1+\delta)S'(r) - \delta\left[B'(r)+C'(r)\right]\right\} > 0$, where the inequality uses $0<\delta<1$, and Lemma A.1 (in Appendix A.2). ■

OA-A.6 Proof of Lemma A.6

The proof proceeds in two parts:

• *Part* (*i*). The derivatives involved in the first-order conditions (A.9) and (A.10) are given by:

$$\begin{aligned}
\frac{dU^{Comp}}{dr}(r) &= 2\left[1-G(r)\right]g(r)(v-r) - G(r), \\
\frac{\partial\bar{\phi}}{\partial r}(r,\delta) &= \left(1-\delta^2\right)\left[G(r)+g(r)(r-\underline{c})\right] - \delta\left[G^2(r) - \delta g(r)\Gamma(r)\right], \\
\frac{\partial\underline{\phi}}{\partial r}(r,\delta) &= \delta\left[\delta G^2(r) - g(r)\Gamma(r)\right],
\end{aligned}$$

where $\Gamma(r) \equiv \int_{\underline{c}}^{r} G(c)dc$. It follows that the first-order conditions (A.9) and (A.10) depend on the cost distribution only through $\{g(r_i^D(\delta)), G(r_i^D(\delta)), \Gamma(r_i^D(\delta))\}_{i=1,2}$.

Let

$$r^D(\delta) \equiv \frac{r_1^D(\delta)+r_2^D(\delta)}{2} \quad \text{and} \quad \Delta^D(\delta) \equiv r_2^D(\delta) - r_1^D(\delta),$$

respectively, denote the mean of and the difference in the optimal deterrence reserves. From Proposition 3, $\Delta^D(\delta) > 0$. Suppose now that $U^{Comp}(r_2^D(\delta)) = U^{Comp}(r_1^D(\delta))$ for some $\delta \in (\delta^C, 1)$, and consider an arbitrary small change in the distribution G that affects $g(\cdot)$, $G(\cdot)$, and $\Gamma(\cdot)$ only in the interval $(r^D(\delta)-\varepsilon, r^D(\delta)+\varepsilon)$, for some $\varepsilon \in (0, \Delta^D(\delta)/2)$, in such a way that:[3]

$$\Delta_\varepsilon^U(\delta) \equiv \int_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon} (v-c)\left[g_\varepsilon(c)G_\varepsilon(c) - g(c)G(c)\right]dc \neq 0,$$

where G_ε and g_ε are the distribution and density associated with the change, respectively.

[1]Recall that $\mathcal{D}(\delta) = \mathcal{D}_S(\delta)\cup\mathcal{D}_1(\delta)\cup\mathcal{D}_2(\delta)$, where $\mathcal{D}_S(\delta)$ and each $\mathcal{D}_i(\delta)$ are compact subsets of $[\underline{c}, \min\{\bar{c}, v\}]^2$. For continuity, see step 3 in Appendix A.3.

[2]Specifically, $r_2^D(\delta) = \hat{r}(r_1^D(\delta),\delta)$, where $\hat{r}(\cdot,\delta)$ is strictly decreasing in δ; it follows that $\mathbf{r}^D(\delta) \notin \mathcal{D}(\delta')$ for any $\delta' > \delta$.

[3]We construct an example of such a change in Online Appendix OA-A.7.

By construction, such a change does not affect $\{g(r_i^D(\delta)), G(r_i^D(\delta)), \Gamma(r_i^D(\delta))\}_{i=1,2}$ (and, thus, does not affect the first-order conditions (A.9) and (A.9)); hence, the optimal deterrence reserves $r_1^D(\delta)$ and $r_2^D(\delta)$ remain unchanged. The payoff $U^{Comp}\left(r_1^D(\delta)\right)$ is also unaffected, as it depends on G only in the range $r \leq r_1^D(\delta) < r - \varepsilon$. By contrast, by altering G in the range $\left(r^D(\delta) - \varepsilon, r^D(\delta) - \varepsilon\right) \subset \left(r_1^D(\delta), r_2^D(\delta)\right)$, the change affects $U^{Comp}\left(r_2^D(\delta)\right)$ by an amount that, using (3), is equal to $\Delta_\varepsilon^U(\delta) \neq 0$. Hence, following the change, the optimal deterrence reserves yield different payoffs in the two markets.

- *Part (ii)*. Fix δ such that $r_2^D(\delta) = r^C$. From (A.10) and (A.11), this amounts to:

$$0 = \frac{\partial \bar{\phi}}{\partial r}(r^C, \delta) = \left(1 - \delta^2\right) S'(r^C) - \delta\left[B'(r^C) - \delta C'(r^C)\right]. \tag{OA-A.1}$$

Because $\frac{\partial^2 \bar{\phi}}{\partial r \partial \delta}(r, \delta) = B'(r) + 2\delta\left[S'(r) - C'(r)\right] > 0$, it follows that, for any $\delta' \neq \delta$, the equality (OA-A.1) is violated, implying that the first-order condition (A.10) cannot be satisfied for $r_2 = r^C$ and $\lambda > 0$. Hence, generically over δ, $r_2^D(\delta) \neq r^C$. ■

OA-A.7 Example of a generic alteration of the cost distribution

We construct here an example of a change in the cost distribution, from G to G_ε, satisfying the conditions required in the proof of Lemma A.6 (see footnote 3 in Section OA-A.7), namely:

(*i*) the change is continuous and affects g, G, and Γ (the primitive of G) in the range $\left(r^D(\delta) - \varepsilon, r^D(\delta) + \varepsilon\right)$ (and only in that range), where

$$r^D(\delta) \equiv \frac{r_1^D(\delta) + r_2^D(\delta)}{2}$$

and ε is an arbitrary number satisfying

$$0 < \varepsilon < \frac{\Delta^D(\delta)}{2},$$

where

$$\Delta^D(\delta)\, r_2^D(\delta) - r_1^D(\delta)$$

denotes the difference in the two deterrence reserves, which is positive from Proposition 3.

(*ii*) the change affecs buyer 2's payoff, which boils down to

$$\Delta_\varepsilon^U(\delta) \equiv \int_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon} (v - c)\left[g_\varepsilon(c) G_\varepsilon(c) - g(c) G(c)\right] dc \neq 0.$$

Consider the following baseline function $f(\cdot)$, defined over $\left[r^D(\delta)-\varepsilon, r^D(\delta)+\varepsilon\right]$:

$$f(c)=\begin{cases} 4+4\frac{c-r^D(\delta)}{\varepsilon} & \text{for } c\in\left[r^D(\delta)-\varepsilon, r^D(\delta)-\frac{3\varepsilon}{4}\right], \\ -2-4\frac{c-r^D(\delta)}{\varepsilon} & \text{for } c\in\left[r^D(\delta)-\frac{3\varepsilon}{4}, r^D(\delta)-\frac{\varepsilon}{4},\right], \\ 4\frac{c-r^D(\delta)}{\varepsilon} & \text{for } c\in\left[r^D(\delta)-\frac{\varepsilon}{4}, r^D(\delta)\right], \\ -4\frac{c-r^D(\delta)}{\varepsilon} & \text{for } c\in\left[r^D(\delta), r^D(\delta)+\frac{\varepsilon}{4}\right], \\ 4\frac{c-r^D(\delta)}{\varepsilon}-2 & \text{for } c\in\left[r^D(\delta)+\frac{\varepsilon}{4}, r^D(\delta)+\frac{3\varepsilon}{4}\right], \\ 4-4\frac{c-r^D(\delta)}{\varepsilon} & \text{for } c\in\left[r^D(\delta)+\frac{3\varepsilon}{4}, r^D(\delta)+\varepsilon\right]. \end{cases}$$

In words, the function $f(\cdot)$ is continuous and, in all segments, its slope is constant and has the same absolute value; furthermore, dividing the interval $\left[r^D(\delta)-\varepsilon, r^D(\delta)+\varepsilon\right]$ into four sub-intervals of equal length, the function $f(\cdot)$ oscillates between -1 and 1 as follows: in the first and last intervals, $\left[r^D(\delta)-\varepsilon, r^D(\delta)-\varepsilon/2\right]$ and $\left[r^D(\delta)+\varepsilon/2, r^D(\delta)+\varepsilon\right]$, it first jumps *up* from 0 to 1, before going back to 0; by contrast, in the two middle intervals $\left[r^D(\delta)-\varepsilon/2, r^D(\delta)\right]$ and $\left[r^D(\delta), r^D(\delta)+\varepsilon/2\right]$, it first jumps *down* from 0 to -1, before going back to 0. It is straightforward to check that this function satisfies (with F denoting the primitive of f, and Φ denoting the primitive of F):

$$\begin{aligned} f\left(r^D(\delta)-\varepsilon\right) &= F\left(r^D(\delta)-\varepsilon\right)=\Phi\left(r^D(\delta)-\varepsilon\right)=0, \\ f\left(r^D(\delta)+\varepsilon\right) &= F\left(r^D(\delta)+\varepsilon\right)=\Phi\left(r^D(\delta)+\varepsilon\right)=0. \end{aligned}$$

It thus satisfies condition (i) above. It follows that any scaled-down function $\rho f(\cdot)$, for any arbitrary small (positive or negative) ρ, also satisfies condition (i)—and for ρ small enough, the modified cost distribution still has a strictly monotone hazard rate.

We now turn to condition (ii). Integrating by parts yields:

$$\begin{aligned} \int_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon}(v-c)g(c)G(c)dc &= \left[(v-c)\frac{G^2(c)}{2}\right]_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon}+\int_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon}\frac{G^2(c)}{2}dc, \\ \int_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon}(v-c)g_\varepsilon(c)G_e(c)dc &= \left[(v-c)\frac{G_\varepsilon^2(c)}{2}\right]_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon}+\int_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon}\frac{G_\varepsilon^2(c)}{2}dc. \end{aligned}$$

It follows that $\Delta_{\varepsilon}^{U}(\delta)$ can be expressed as, for $G_{\varepsilon}(\cdot) = G(\cdot) + \rho F(\cdot)$:

$$\begin{aligned}
\Delta_{\varepsilon}^{U}(\delta) &= \frac{1}{2}\int_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon} \left[G_{\varepsilon}^{2}(c) - G^{2}(c)\right] dc \\
&= \frac{1}{2}\int_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon} \left\{[G(c) + \rho F(c)]^{2} - G^{2}(c)\right\} dc \\
&= \rho\int_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon} G(c)F(c)\, dc + \frac{1}{2}\int_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon} F^{2}(c)\, dc,
\end{aligned}$$

where the first equality stems from $G_{e}\left(r^{D}(\delta) - \varepsilon\right) = G\left(r^{D}(\delta) - \varepsilon\right)$ and $G_{e}\left(r^{D}(\delta) + \varepsilon\right) = G\left(r^{D}(\delta) + \varepsilon\right)$, implying that the bracketed terms coincide in the previous expressions. In the last expression, the second term is positive. Hence, if

$$\int_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon} G(c)F(c)\, dc \geq 0,$$

we have $\Delta_{\varepsilon}^{U}(\delta) > 0$. If instead

$$\int_{r^D(\delta)-\varepsilon}^{r^D(\delta)+\varepsilon} G(c)F(c)\, dc < 0,$$

then $\Delta_{\varepsilon}^{U}(\delta) > 0$ for any $\rho < 0$. Hence, in both cases there exists a change satisfying condition (ii).

OA-B Extension to more than two markets

In this section, we consider an extension that allows for $n \geq 2$ markets. We continue to assume that there are two suppliers and focus on the case with one integrated buyer.

OA-B.1 Setting

Let $\mathcal{N} \equiv \{1, \ldots, n\}$ denote the set of markets and $\mathbf{r} = (r_1, \ldots, r_n)$ the vector of reserves in these markets, with the convention that markets are labeled by decreasing order of the reserves: that is, $r_1(n) \geq \cdots \geq r_n(n)$. As before, each market is characterized by the same value v for the buyer and the same distribution G over $[\underline{c}, \overline{c}]$ for the sellers' constant marginal costs, where cost draws are independent across suppliers and time.

Because symmetry facilitates collusion, for the sake of exposition we focus on market allocation that are as balanced as possible. Hence, if n is even, then each supplier is the designated winner in $n/2$ markets, alternating each period. For example, with $n = 4$, the markets might be divided up as $\{1, 2\}$ and $\{3, 4\}$, with supplier 1 designated for $\{1, 2\}$ in one period and for $\{3, 4\}$ in the next period. If n is odd, then suppliers alternate being the designated winner in $(n+1)/2$ and in $(n-1)/2$ markets. For example with $n = 3$, the markets might be divided up as $\{1, 2\}$ and $\{3\}$, with supplier 1 designated for $\{1, 2\}$ in one period and for $\{3\}$ in the next period.[4]

Consider a supplier facing the lowest cost and designated for the markets *other than* $\mathcal{M} \subset \mathcal{N}$. By deviating, the supplier can get the monopoly payoff rather than the non-designated supplier payoff in all markets in $\mathcal{M}$; the associated short-term stake is thus:

$$S_{\mathcal{M}}(\mathbf{r}) \equiv \sum_{j \in \mathcal{M}} S(r_j).$$

Although $S_{\mathcal{M}}(\mathbf{r})$ only depends on $(r_j)_{j \in \mathcal{M}}$, it is notationally convenient to write it as a function of the entire vector $\mathbf{r}$.

The long-term stake for a supplier that would forever be designated for the markets in $\mathcal{M}$ is instead given by

$$L_{\mathcal{M}}(\mathbf{r}, \delta) \equiv \frac{\delta}{1-\delta} \left[\sum_{i \in \mathcal{M}} B(r_i) - \sum_{j \in \mathcal{N} \setminus \mathcal{M}} C(r_j) \right],$$

where $B(\cdot)$ and $C(\cdot)$ denote the benefit and cost of collusion, given by (1) and (2). Then the long-term stake for a supplier that is designated for the markets in $\mathcal{M}$ next period, accounting for the rotation over the set of designated markets, is

$$L^R_{\mathcal{M}}(\mathbf{r}, \delta) \equiv \frac{1}{1+\delta} L_{\mathcal{M}}(\mathbf{r}, \delta) + \frac{\delta}{1+\delta} L_{\mathcal{N} \setminus \mathcal{M}}(\mathbf{r}, \delta).$$

Define k to be half the number of markets if there is an even number of markets and that number rounded up if there is an odd number of markets:

$$k \equiv \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Collusion is not incentive compatible if a supplier has an incentive to deviate in k markets when it is designated for $n - k$ markets. Thus, given reserves $\mathbf{r}$, the suppliers are deterred from collusion if and only if for all $\mathcal{M} \in \mathcal{P}(\mathcal{N}, k)$, where $\mathcal{P}(\mathcal{N}, k)$ is the set of permutations of subsets of $\mathcal{N}$ containing k elements, either $L^R_{\mathcal{M}}(\mathbf{r}, \delta) \leq S_{\mathcal{M}}(\mathbf{r})$ or $L^R_{\mathcal{N} \setminus \mathcal{M}}(\mathbf{r}, \delta) \leq S_{\mathcal{N} \setminus \mathcal{M}}(\mathbf{r})$. For example, if $n = 4$, then a market allocation in which each supplier alternates between

[4] It can be shown that these market allocations do indeed maximize the scope for collusion for the optimal accommodation and deterrence reserves.

markets $\{1,2\}$ and $\{3,4\}$ is deterred if either

$$L^R_{\{1,2\}}(\mathbf{r},\delta) \leq S_{\{1,2\}}(\mathbf{r}) \;\text{ or }\; L^R_{\{3,4\}}(\mathbf{r},\delta) \leq S_{\{3,4\}}(\mathbf{r}).$$

OA-B.2 Optimal reserves

The buyer's optimal reserves conditional on deterrence satisfy:

$$\max_{\mathbf{r}} \sum_{i\in\mathcal{N}} U^{Comp}(r_i)$$

subject to, for all $\mathcal{M} \in \mathcal{P}(\mathcal{N},k)$, either $L^R_{\mathcal{M}}(\mathbf{r},\delta) \leq S_{\mathcal{M}}(\mathbf{r})$ or $L^R_{\mathcal{N}\setminus\mathcal{M}}(\mathbf{r},\delta) \leq S_{\mathcal{N}\setminus\mathcal{M}}(\mathbf{r})$. The buyer then compares this payoff to $nU^{Coll}(r^{Coll})$ to determine whether to accommodate or deter collusion.

We first note that, as long as the number of markets remains even, asymmetric reserves still help to reduce the cost of deterrence:

Proposition OA-B.1. *If the number of markets is even and collusion is not blockaded, then an integrated buyer's optimal deterrence reserves are asymmetric.*

Proof. Suppose that there are $n = 2k$ markets. We first show that, for symmetric reserves, the scope for collusion is maximized when each supplier is designated for half of the markets. Let r denote the symmetric reserve and suppose without loss of generality that a supplier is currently designated for $n-h$ markets, for some $h \in \mathcal{N}$. The supplier's short-term stake from a deviation in the remaining h markets is then given by

$$S\left(r,h\right) \equiv hS\left(r\right),$$

whereas its long-term stake is:

$$L\left(r,h,\delta\right) \equiv \frac{\delta\left[hB\left(r\right)-\left(n-h\right)C\left(r\right)\right]+\delta^2\left[\left(n-h\right)B\left(r\right)-hC\left(r\right)\right]}{1-\delta^2}.$$

Hence, the supplier has no incentive to deviate if $\phi(r,h,\delta) \geq 0$, where

$$\begin{aligned}\phi(r,h,\delta) &\equiv \left(1-\delta^2\right)\left[L(r,h,\delta)-S\left(r,h\right)\right]\\ &= \delta\left[hB\left(r\right)-\left(n-h\right)C\left(r\right)\right]+\delta^2\left[\left(n-h\right)B\left(r\right)-hC\left(r\right)\right]-\left(1-\delta^2\right)hS\left(r\right),\end{aligned}$$

which is decreasing in h, that is,

$$\frac{\partial\phi(r,h,\delta)}{\partial h} = \left(1-\delta\right)\left[\delta B\left(r\right)+\delta C\left(r\right)-\left(1+\delta\right)S\left(r\right)\right] < 0,$$

where the inequality stems from Lemma A.1. Collusion is sustainable if no supplier has an incentive to deviate, that is, if

$$\min\{\phi(r,h,\delta),\phi(r,n-h,\delta)\}\geq 0.$$

It follows that collusion is easiest to sustain when $h=n-h=k(=n/2)$. In particular, collusion is blockaded if $\phi\left(r^C,k,\delta\right)\geq 0$.

Suppose now that collusion is not blockaded. Because the buyer's payoff $U^{Comp}(r)$ is concave in r, the optimal symmetric deterrence reserve, $r_S^D(\delta)$, is then such that $\phi\left(r_S^D(\delta),k,\delta\right)=0$. Starting from $\mathbf{r}_S^D(\delta)=\left(r_S^D(\delta),...,r_S^D(\delta)\right)$, consider now a small change in reserves in which r_1 is slightly increased by $dr_1=(n-1)\,dr>0$, whereas all other reserves are reduced by dr. By construction, this small change in the reserves has no first-order effect on the buyer's overall payoff, as the net impact is given by

$$\sum_{i\in\mathcal{N}}\frac{\partial U^{Comp}}{\partial r_i}(r_i)\,dr_i\bigg|_{r_i=r_S^D(\delta)}=\frac{\partial U^{Comp}}{\partial r}(r)\bigg|_{r=r_S^D(\delta)}[dr_1-(n-1)\,dr]=0.$$

However, for the supplier currently *not* designated for market 1, the short-term stake becomes (where $r\equiv r_S^D(\delta)-dr$)

$$\hat{S}(\mathbf{r})\equiv S(r_1)+(k-1)\,S(r)\,,$$

whereas its long-term stake is

$$\hat{L}^R(\mathbf{r},\delta)\equiv\frac{\delta\left[B(r_1)+(k-1)\,B(r)-kC(r)\right]+\delta^2\left[kB(r)-C(r_1)-(k-1)\,C(r)\right]}{1-\delta^2}.$$

The supplier thus has an incentive to deviate if $\hat{\phi}(\mathbf{r},\delta)\equiv\left(1-\delta^2\right)\left[\hat{L}^R(\mathbf{r},\delta)-\hat{S}(\mathbf{r})\right]<0.$

A first-order approximation yields:

$$
\begin{aligned}
\hat{\phi}(\mathbf{r},\delta) &\simeq \hat{\phi}\left(\mathbf{r}_S^D(\delta),\delta\right)+ \\
&\quad \delta\left\{B'\left(r_S^D(\delta)\right)\left[dr_1-(k-1)\,dr\right]-kC'\left(r_S^D(\delta)\right)(-dr)\right\} \\
&\quad +\delta^2\left\{kB'\left(r_S^D(\delta)\right)(-dr)-C'\left(r_S^D(\delta)\right)\left[dr_1-(k-1)\,dr\right]\right\} \\
&\quad -\left(1-\delta^2\right)S'\left(r_S^D(\delta)\right)\left[dr_1-(k-1)\,dr\right] \\
&= \delta\left[B'\left(r_S^D(\delta)\right)+C'\left(r_S^D(\delta)\right)\right]kdr \\
&\quad -\delta^2\left[B'\left(r_S^D(\delta)\right)+C'\left(r_S^D(\delta)\right)\right]kdr \\
&\quad -\left(1-\delta^2\right)S'\left(r_S^D(\delta)\right)kdr \\
&= (1-\delta)\left\{\delta\left[B'\left(r_S^D(\delta)\right)+C'\left(r_S^D(\delta)\right)-(1+\delta)\,S'(r)\right]\right\}kdr \\
&< 0,
\end{aligned}
$$

where the first equality follows from $\hat{\phi}\left(\mathbf{r}_S^D(\delta),\delta\right)=\phi\left(r_S^D(\delta),k,\delta\right)=0$ and $dr_1=(2k-1)dr$, whereas the inequality stems from $\delta\in(0,1)$, $dr>0$ and Lemma A.1. It follows that the change in reserves strictly deters collusion while maintaining the buyer's total payoff. By continuity, there exists a neighboring change in reserves that keeps deterring collusion and enhances the buyer's payoff. ■

To go further, we now focus on the case in which $v=1$ and costs are uniformly distributed over $[0,1]$. Table OA-B.2 reports the buyer's optimal reserve policy for different numbers of markets (from $n=1$ to $n=6$) and a given value of the discount factor ($\delta=0.94$).

Table OA-B.2: Optimal deterrence reserves $\mathbf{r}^D$ for $\delta=0.94$

n	n even	n odd
1		$r_1=0.5$ (blockaded)
2	$r_1=0.4894$, $r_2=0.3937$	
3		$r_1=\cdots=r_3=0.4953$
4	$r_1=0.4849$, $r_2=\cdots=r_4=0.3894$	
5		$r_1=\cdots=r_5=0.4512$
6	$r_1=0.4834$, $r_2=\cdots=r_6=0.3875$	

Note: Assumes $v=1$ and uniformly distributed costs.

Several features can be noted. First, for even numbers of markets, the asymmetry established by Proposition OA-B.1 takes a specific form, where a single reserve is set above the others. Intuitively, treating $n-1$ markets equally enhances the buyer's expected payoff because $U^{Comp}(r)$ is concave in r, and also limits the suppliers' ability to restore symmetry by optimizing over the composition of designated packages.[5] Second, for odd numbers of markets, the optimal reserve policy is instead symmetric.[6] This is because the market allocation itself is necessarily imbalanced (with one supplier designated for $(n+1)/2$ and the other for $(n-1)/2$ markets), to an extent such that there is no need to introduce further asymmetry.[7] Third, for each type of situation, collusion becomes easier as the number of markets increases, which in turn calls for more aggressive reserves. That is, letting $\mathbf{r}^D(n) = (r_1^D(n), \ldots, r_n^D(n))$ denote the optimal deterrence reserves, we have $\mathbf{r}^D(n+2) < \mathbf{r}^D(n)$. To see why, consider first the case of even numbers of markets. If the buyer were restricted to symmetric reserves, then increasing the number of markets would raise proportionally the short-term and long-term stakes, and thus have no impact on the scope for collusion.[8] However, the buyer finds it optimal to introduce an asymmetry by setting one reserve above the others and, given the optimal level of asymmetry, to adjust the overall level of the reserves so as to ensure that the supplier not designated for that market has an incentive to deviate. As the number of markets increases, however, the long-term stake increases proportionally, which in turn calls for more aggressive reserves.

For odd numbers of markets, the optimal deterrence reserves are symmetric (at least for up to six markets), as the market allocation is itself sufficiently asymmetric. However, as the number of markets increases, the relative asymmetry of the market allocation is reduced, which calls again for more aggressive reserves.

[5] For instance, if $n = 4$ and $r_1 > r_2 > r_3 > r_4$, the suppliers can maintain some symmetry by designating one supplier for markets 1 and 4 and the other for markets 2 and 3. By contrast, with $r_1 > r_2 = r_3 = r_4$, one supplier necessarily ends up with a better designated packaged and the buyer moreover perfectly controls the level of asymmetry.

[6] It could therefore be implemented as well by independent buyers, the symmetry of the optimal reserves ensuring that the preference for accommodation versus deterrence is the same for all buyers, integrated or not—and conditional on deterrence, the optimal reserves constitute a Nash equilibrium of the reserve-setting game.

[7] As the number of markets increases, this imbalance however tends to become relatively small; asymmetric reserves may thus become again optimal.

[8] The optimal symmetric reserve is $\hat{\delta}^{-1}(\delta)$, where the condition determining the threshold $\hat{\delta}(\cdot)$ is given by $\frac{\hat{\delta}(r)}{1-\hat{\delta}(r)} = \frac{S(r)}{B(r)-C(r)}$, as increasing the number of markets from 2 to $n = 2k$ leads to multiply both the numerator and denominator of the right-hand side by k.

Table OA-B.3: Threshold discount factor between deterrence and accommodation (δ^A)

n	n even	n odd
1		0.9714
2	0.9541	
3		0.9586
4	0.9501	
5		0.9545
6	0.9489	

Note: Assumes $v = 1$ and uniformly distributed costs.

It follows from the last observation that, for each type of situation (i.e., even or odd number of markets), deterrence becomes more costly as the number of markets increases, and is thus less likely to be optimal. This intuition is confirmed by Table OA-B.3, which reports, for the same numbers of markets as before, the discount factor threshold $\delta^A(n)$ above which accommodation dominates deterrence. We have:

$$\delta^A(n+2) < \delta^A(n).$$

Thus, as the number of markets increases, deterrence is optimal for a smaller range of discount factors.

In contrast, increasing the number of markets from an *even* to an *odd* number introduces an intrinsic asymmetry in the market allocation and can make collusion more fragile, and thus easier to deter. Indeed, deterrence is optimal for a wider range of discounts factors with $n = 3$ or even with $n = 5$ than with $n = 2$.

OA-C Illustration of monotonicity assumptions

In this section, we show that our monotonicity assumptions are satisfied when costs are distributed over $[0, 1]$ according to the power distribution $G(c) = c^{1/s}$ with $s > 0$ and $v \geq 1$. Specifically, we show that the unique-market discount factor threshold, $\hat{\delta}_U(r)$ is decreasing in r.

For this setup, we have:

$$B(r) = \int_0^r G^2(c)\,dc = \frac{sr^{1+\frac{2}{s}}}{2+s}, \tag{OA-C.2}$$

$$C(r) = \int_0^r [G(r) - G(c)]\,G(c)\,dc = \frac{sr^{1+\frac{2}{s}}}{(1+s)(2+s)}, \tag{OA-C.3}$$

$$S(r) = G(r)(r - \underline{c}) = r^{1+\frac{1}{s}}.$$

We first show that the critical discount factors thresholds $\hat{\delta}_S(r) = \hat{\delta}(r,r)$ and $\hat{\delta}_U(r) = \hat{\delta}(\underline{c}, r)$ are decreasing in r, before turning to the monotonicity of the long-term stake. For symmetric reserves equal to r, the threshold $\hat{\delta}_S(r)$, given by $(A.1)$, is equal to

$$\hat{\delta}_S(r) = \frac{1}{1+\frac{B(r)-C(r)}{S(r)}} = \frac{1}{1+\frac{s^2 r^{\frac{1}{s}}}{(1+s)(2+s)}},$$

which is strictly decreasing in r over the relevant range $r \in [0,1]$. For a unique market, the threshold $\hat{\delta}_U(r)$, given by $(A.2)$, is equal to

$$\begin{aligned}\hat{\delta}_U(r) &= \sqrt{\frac{S(r)}{S(r)-C(r)} + \frac{B^2(r)}{4[S(r)-C(r)]^2}} - \frac{B(r)}{2[S(r)-C(r)]} \\ &= \sqrt{\frac{(1+s)(2+s)}{(1+s)(2+s)-sr^{\frac{1}{s}}} + \left[\frac{1+s}{2}\frac{sr^{\frac{1}{s}}}{(1+s)(2+s)-sr^{\frac{1}{s}}}\right]^2} - \frac{1+s}{2}\frac{sr^{\frac{1}{s}}}{(1+s)(2+s)-sr^{\frac{1}{s}}}.\end{aligned}$$

Using

$$x(r) \equiv \frac{1+s}{2}\frac{sr^{\frac{1}{s}}}{(1+s)(2+s)-sr^{\frac{1}{s}}}, \tag{OA-C.4}$$

this threshold can be expressed as $\hat{\delta}_U(r) = \delta_U(x(r))$, where:

$$\delta_U(x) \equiv \sqrt{1+\frac{2x}{1+s}+x^2} - x \tag{OA-C.5}$$

is strictly decreasing in x:

$$\delta_U'(x) = \frac{\frac{1}{1+s}+x}{\sqrt{1+\frac{2z}{1+s}+x^2}} - 1 = \frac{\sqrt{\frac{1}{(1+s)^2}+\frac{2x}{1+s}+x^2}}{\sqrt{1+\frac{2x}{1+s}+x^2}} - 1 < 0.$$

Because $x(r)$ is strictly increasing in r, it follows that $\delta_U(x)$ is strictly decreasing in x.

We now show that the long-term stake $L(r_j, r_i, \delta)$ is strictly increasing in r_j the relevant range $\delta > \underline{\delta} \equiv \inf_{\mathbf{r}\in[\underline{c},\min\{v,\overline{c}\}]^2}\hat{\delta}(\mathbf{r})$. We have:

$$\frac{\partial L(r_j, r_i, \delta)}{\partial r_j} = \frac{\delta}{1-\delta^2}\left[\delta B'(r_j) - C'(r_j)\right].$$

It follows that $L(r_j, r_i, \delta)$ is strictly increasing in r_j if and only if $\delta B'(r_j) > C'(r_j)$, which amounts to

$$\delta > \frac{C'(r_j)}{B'(r_j)}.$$

From $(OA-C.2)$ and $(OA-C.3)$, the right-hand side is constant and equal to:

$$\frac{B(r)}{C(r)}=\frac{\frac{sr^{1+\frac{2}{s}}}{(1+s)(2+s)}}{\frac{sr^{1+\frac{2}{s}}}{2+s}}=\frac{1}{1+s}.$$

To conclude the argument, we now show that $\underline{\delta}>1/(1+s)$. The argument relies on four steps.

• *Step* 1. *For any* $r\in[\underline{c},\min\{v,\overline{c}\}]$, $\hat{\delta}(r,r)>1/(1+s)$. Fix $r\in[\underline{c},\min\{v,\overline{c}\}]$. From the above observations, the threshold $\hat{\delta}(r,r)=\hat{\delta}_S(r)$ is strictly decreasing in r; furthermore, for $r=1$ it is equal to

$$\hat{\delta}_S(1)=\frac{1}{1+\frac{s^2}{(1+s)(2+s)}}>\frac{1}{1+s}.$$

The conclusion follows.

• *Step* 2. *For any* $r\in[\underline{c},\min\{v,\overline{c}\}]$, $\hat{\delta}(\underline{c},r)>1/(1+s)$. Fix $r\in[\underline{c},\min\{v,\overline{c}\}]$. From the above observations, the threshold $\hat{\delta}(\underline{c},r)=\hat{\delta}_U(r)$ can be expressed as $\delta_U(x)$, for $x=x(r)\,(\geq 0)$ given by $(OA-C.4)$. Furthermore,

$$\begin{aligned}\delta_U(x)>\frac{1}{1+s}\quad&\Longleftrightarrow\quad\sqrt{1+\frac{2x}{1+s}+x^2}>x+\frac{1}{1+s}\\&\Longleftrightarrow\quad 1+\frac{2x}{1+s}+x^2>\left(x+\frac{1}{1+s}\right)^2\\&\Longleftrightarrow\quad 1>\frac{1}{(1+s)^2},\end{aligned}$$

where the first equivalence stems from $(OA-C.5)$ and the second one from $x>0$ (ensuring that $1+\frac{2x}{1+s}+x^2$ and $x+\frac{1}{1+s}$ are both positive), and the last inequality holds trivially as $s>0$. The conclusion follows.

• *Step* 3. *For any* $r\in[\underline{c},\min\{v,\overline{c}\}]$ *and any* $\tilde{r}\in[\underline{c},r]$, $\hat{\delta}(r,\tilde{r})$ *is weakly decreasing in* $\tilde{r}$. Fix $r\in[\underline{c},\min\{v,\overline{c}\}]$ and $\tilde{r}\in[\underline{c},r]$. From Lemma 2, $L(\tilde{r},r,\delta)$ is strictly increasing in δ and $\hat{\delta}(r,\tilde{r})$ is the unique solution in δ to

$$L(\tilde{r},r,\delta)=S(r).$$

As $L(\tilde{r},r,\delta)$ is twice continuously differentiable in δ and $\tilde{r}$, $\hat{\delta}(r,\tilde{r})$ is continuously differentiable in $\tilde{r}$ and:

$$\left.\frac{\partial\hat{\delta}(r_j,r)}{\partial r_j}\right|_{r_j=\tilde{r}}=-\frac{\left.\frac{\partial L(r_j,r,\delta)}{\partial r_j}\right|_{r_j=\tilde{r},\delta=\hat{\delta}(r,\tilde{r})}}{\left.\frac{\partial L(r_j,r,\delta)}{\partial\delta}\right|_{r_j=\tilde{r},\delta=\hat{\delta}(r,\tilde{r})}},$$

where $\partial L(r_j, r, \delta)/\partial\delta|_{r_j=\tilde{r},\delta=\hat{\delta}(r,\tilde{r})} > 0$ and:

$$\left.\frac{\partial L(r_j, r, \delta)}{\partial r_j}\right|_{r_j=\tilde{r},\delta=\hat{\delta}(r,\tilde{r})} = \frac{\hat{\delta}(r,\tilde{r})}{1-\hat{\delta}^2(r,\tilde{r})}\left[\hat{\delta}(r,\tilde{r})B'(\tilde{r}) - C'(\tilde{r})\right] = \frac{\hat{\delta}(r,\tilde{r})B'(\tilde{r})}{1-\hat{\delta}^2(r,\tilde{r})}\left[\hat{\delta}(r,\tilde{r}) - \frac{1}{1+s}\right].$$

It follows that:

$$\left.\frac{\partial\hat{\delta}(r_j, r)}{\partial r_j}\right|_{r_j=\tilde{r}} \lesseqgtr 0 \iff \left.\frac{\partial L(r_j, r, \delta)}{\partial r_j}\right|_{r_j=\tilde{r},\delta=\hat{\delta}(r,\tilde{r})} \gtreqless 0 \iff \hat{\delta}(r,\tilde{r}) \gtreqless \frac{1}{1+s}. \qquad \text{(OA-C.6)}$$

Suppose now by way of contradiction that $\partial\hat{\delta}(r_j, r)/\partial r_j\Big|_{r_j=\tilde{r}} > 0$ for some $\tilde{r} \in (\underline{c}, r]$, and let $\underline{\tilde{r}} \equiv \inf\left\{\check{r} \in [\underline{c}, \tilde{r}] \mid \partial\hat{\delta}(r_j, r)/\partial r_j\Big|_{r_j=\check{r}} > 0\right\}$. From $(OA - C.6)$, $\hat{\delta}(r, \check{r}) < 1/(1+s)$ for any $\check{r} \in (\underline{\tilde{r}}, \tilde{r}]$. Furthermore, from step 2, $\hat{\delta}(r, \underline{c}) > 1/(1+s)$. Hence, $\underline{\tilde{r}} > \underline{c}$ and, by continuity, $\hat{\delta}(r, \underline{\tilde{r}}) = 1/(1+s) > \hat{\delta}(r, \tilde{r})$. It follows that $\check{r} \in (\underline{\tilde{r}}, \tilde{r}]$ such that $\hat{\delta}(r, \check{r}) < 1/(1+s)$ (by continuity) and $\partial\hat{\delta}(r_j, r)/\partial r_j\Big|_{r_j=\check{r}} < 0$ (by definition of $\underline{\tilde{r}}$), contradicting $(OA - C.6)$. It follows that $\partial\hat{\delta}(r_j, r)/\partial r_j\Big|_{r_j=\tilde{r}} \leq 0$ $\hat{\delta}(r, \tilde{r}) \geq 1/(1+s)$ for any $\tilde{r} \in [\underline{c}, r]$.

- *Step* 4. $\underline{\delta} > 1/(1+s)$. Fix $\mathbf{r} = (r_1, r_2)$ and let $\bar{r} \equiv \max\{r_1, r_2\}$. We have:

$$\hat{\delta}(\mathbf{r}) \geq \hat{\delta}(\bar{r}, \bar{r}) = \hat{\delta}_S(\bar{r}) \geq \hat{\delta}_S(1),$$

where the first inequality stems from step 3 and the symmetry of $\hat{\delta}(\cdot)$ (namely, $\hat{\delta}(r_1, r_2) = \hat{\delta}(r_2, r_1)$), and the second one stems from the monotonicity of $\hat{\delta}_S(r)$. Hence:

$$\underline{\delta} = \hat{\delta}_S(1) = \frac{2+3s+s^2}{2+3s+2s^2} = \frac{1}{1+\frac{s^2}{2+3s+s^2}} > \frac{1}{1+s}$$

where the inequality stems from $s < 2+3s+s^2$.

It follows from the above that, for any $\delta \geq \underline{\delta}$, the long-term stake $L(r_j, r_i, \delta)$ is strictly increasing in r_j. This, in turn, implies that the threshold $\hat{\delta}(r_i, r_j)$ is strictly increasing in r_j in the range $r_j \leq r_i$.

Example: uniform distribution. For $s = 1$, we have:

$$\frac{C'(\cdot)}{B'(\cdot)} = \frac{1}{2} < \underline{\delta} = \frac{6}{7},$$

and:

$$\hat{\delta}_U(r) = \frac{\sqrt{36 - 6r + r^2} - r}{6 - r} \text{ and } \hat{\delta}_S(r) = \frac{6}{6 + r},$$

which, as r increases, strictly decrease from $\hat{\delta}_S(0) = \hat{\delta}_U(0) = 1$ to, respectively, $\hat{\delta}_U(1) = \left(\sqrt{31} - 1\right)/5 \simeq 0.91$ and $\hat{\delta}_S(1) = \underline{\delta} = 6/7 \simeq 0.86$.

OA-D Illustrations of coordination and coordination failure with independent buyers

To illustrate two ways in which one obtains no coordination failure, Figure D.1 considers the example in which costs are uniformly distributed over $[0, 1]$ and $v = 1$. The panels depict the deterrence boundary and the buyers' best-responses for different values of the discount factor. Interestingly, the scope for coordination failure is not monotonic in the discount factor. From Proposition 4, when the discount factor is sufficiently high that an integrated buyer accommodates collusion, $\delta > \delta^A$, then the unique Nash equilibrium of the reserve-setting game also involves accommodation, as illustrated in Figure D.1(a). At the other extreme, as illustrated in Figure D.1(b), when the discount factor is sufficiently low that collusion is blockaded, $\delta < \delta^C$, the unique Nash equilibrium of the reserve-setting game has independent buyers both setting a reserve of r^C, just as an integrated buyer would do.

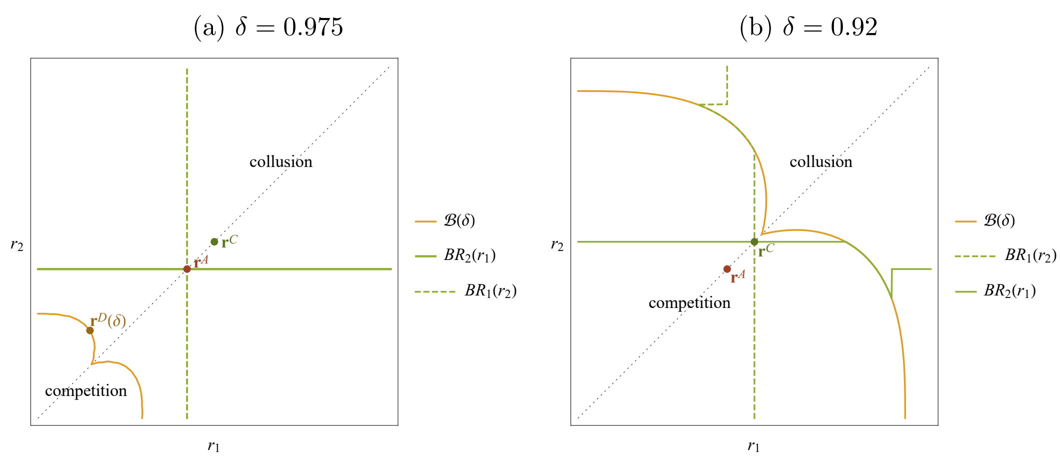

Figure D.1: No coordination failure: an integrated buyer's optimal reserves, $\mathbf{r}^A$ in the case of panel (a) and $\mathbf{r}^C$ in the case of panel (b), are the unique Nash equilibrium of the reserve-setting game. The panels depict the deterrence boundaries and the buyers best-responses over the full relevant range $r_i \in [0, 1]$. Assumes that costs are uniformly distributed over $[0, 1]$, $v = 1$, and δ is as indicated. In this setup, $\delta^C = 0.9231$ and $\delta^A = 0.9540$, so an integrated buyer accommodates collusion in panel (a), and collusion is blockaded in panel (b).

There exists a threshold discount factor $\delta_N^D \in (\delta^C, \delta^A)$ such that coordination failure arises for sure, that is, for $\delta \in (\delta_N^D, \delta^A)$, an integrated buyer would deter collusion using the optimal deterrence reserves, but those optimal deterrence reserves do not constitute a Nash equilibrium of the reserve-setting game with independent buyers.[9] For instance, in Figure 3(a) in the body of the paper, which has $\delta \in (\delta_N^D, \delta^A)$ but close to δ^A, the only Nash equilibrium involves accommodation. Considering a lower δ, but still in the range (δ_N^D, δ^A), Figure D.2(a) shows a case in which there exists an accommodation equilibrium and also a continuum of deterrence equilibria, all of which are suboptimal. For still lower δ, Figure D.2(b) shows a case in which there only exist deterrence equilibria, all of which are suboptimal.

(a) $\delta = 0.94827$ (b) $\delta = 0.94800$ (detail)

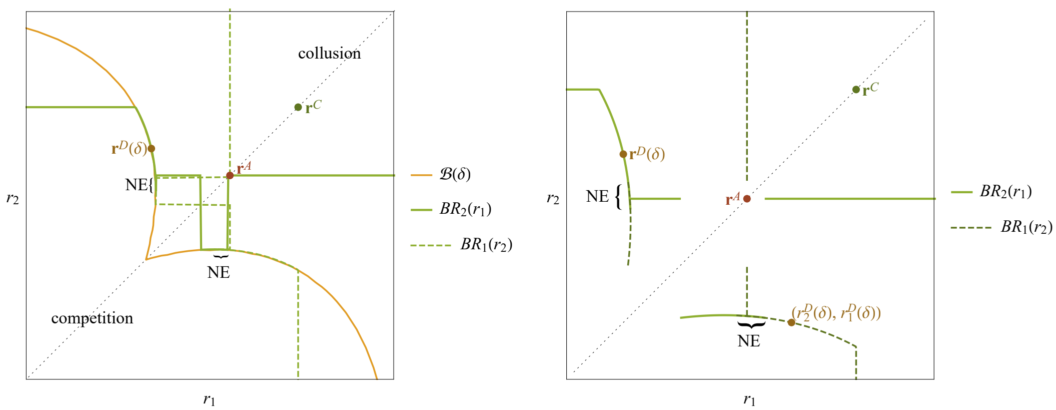

Figure D.2: Coordination failure: an integrated buyer deters collusion with optimal deterrence reserves $\mathbf{r}^D(\delta)$, but in the reserve-setting game with independent buyers, those optimal reserves are not a Nash equilibrium. Panel (a) depicts the deterrence boundaries, buyers' best-responses, and the diagonal; but to reduce clutter, panel (b) shows only the best responses and diagonal. Both panels assume that costs are uniformly distributed over $[0, 1]$ and $v = 1$. The discount factor δ is as indicated above the panels. In this setup, $\delta^C = 0.9231$ and $\delta^A = 0.9540$, so we have $\delta \in (\delta^C, \delta^A)$.

[9] In the examples of Figure 3 in the body of the paper and Figure D.2 here, we have $\delta^C = 0.9231$, $\delta_N^D = 0.9475$, and $\delta^A = 0.9540$.